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Two-Loop $N_F = 1$ QED Bhabha Scattering: Soft Emission and Numerical Evaluation of the Differential Cross-section

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Abstract

Recently, we evaluated the virtual cross-section for Bhabha scattering in pure QED, up to corrections of order $\alpha^4(N_F = 1)$. This calculation is valid for arbitrary values of the squared center of mass energy s and momentum transfer t ; the electron and positron mass m was considered a finite, non vanishing quantity. In the present work, we supplement the previous calculation by considering the contribution of the soft photon emission diagrams to the differential cross-section, up to and including terms of order $\alpha^4(N_F = 1)$. Adding the contribution of the real corrections to the renormalized virtual ones, we obtain an UV and IR finite differential cross-section; we evaluate this quantity numerically for a significant set of values of the squared center of mass energy s .

Key words: Feynman diagrams, Multi-loop calculations, Box diagrams, Bhabha scattering

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1 Introduction

The relevance of the Bhabha scattering process ($e^+e^- \rightarrow e^+e^-$) in the study of the phenomenology of particle physics can hardly be overestimated. This is due to the fact that Bhabha scattering is the process employed to determine the luminosity of all the present $e^+ - e^-$ colliders, at both high (~ 100 GeV) and intermediate ($\sim 1-10$ GeV) energies, as well as in a future linear collider. For this reason, Bhabha scattering has been studied in great detail within the context of the electroweak Standard Model (we refer the interested reader to [1] and references therein).

In recent years, it was pointed out that, while the one-loop radiative corrections to Bhabha scattering and the corresponding real emission corrections were well known within the full electroweak Standard Model [2, 3], the two-loop corrections had not been calculated even in pure QED, although a large amount of work was devoted to the study of the contributions enhanced by factors of $\ln(s/m_e^2)$ [4]. The reason for this situation was identified in the technical problem of calculating the necessary two-loop box diagrams. Several groups started to work on different aspects of the problem. In [5], the two-loop QED virtual cross-section in the limit of zero electron mass was calculated, while in [6] the IR divergent structure of that result was carefully studied. A very interesting and ambitious project aiming at the complete evaluation of the Bhabha scattering cross-section in QED, without neglecting the electron mass m , was presented in [7]. Some of the necessary Feynman diagrams were calculated in [8, 9, 10].

By employing the results of [8, 9], it was possible to complete the calculation of the virtual Bhabha scattering unpolarized differential cross-section, including the contribution of two-loop graphs involving a closed fermion loop (conventionally indicated as corrections of order $\alpha^4(N_F = 1)$, where α is the fine structure constant) [11]. The cross section presented in [11] is valid for arbitrary values of the squared center of mass energy s and momentum transfer t . The full dependence of the cross-section on the electron and positron mass m was retained. In calculating the Feynman diagrams, both UV and IR divergences were regularized with the continuous D -dimensional regularization scheme [12], while the diagrams were evaluated analytically in [8, 9] by means of the Laporta algorithm [13] and the differential equations technique for the evaluation of the master integrals [14]. In [11], the renormalization program was carried out in the on-shell scheme; therefore the final result, expressed in terms of 1- and 2-dimensional harmonic polylogarithms (HPLs, 2dHPLs) [15] of maximum weight 3, is free from UV divergences. However, the cross-section of [11] still includes IR divergent terms that appear as a pole in $(D - 4)$.

As is well known, the IR divergent term in the cross-section of [11] cancels if one adds to it the contribution of events of the type $e^+e^- \rightarrow e^+e^-\gamma$, in the limit in which the photon in the final state carries an energy which is small with respect to the squared center of mass energy. Such events are commonly referred to as *soft* photon events, and their contributions to the cross-section are known as *real* (as opposed to virtual) corrections.

The purpose of the present paper is to calculate the contribution of the the soft radiation diagrams (up to and including order $\alpha^4(N_F = 1)$) to the Bhabha scattering

differential cross-section, in order to verify that the IR divergent terms present in this contribution cancel against the IR divergent terms in the cross-section in [11], as well as to ultimately obtain the IR finite differential cross-section by adding the virtual and real corrections.

In calculating the contribution of the soft photon emission diagrams to the cross-section, we integrated over the soft photon phase space, taking into consideration the emission of photons with an energy smaller than a certain energy threshold ω . This threshold is supposed to be small with respect to the beam energy E . In other words, we assumed that the idealized detector, to be used to measure the differential cross-section we calculated, can tag all the events in which one photon appears in the final state, provided that the photon carries an energy larger than the chosen threshold. We also assumed that all the events in which a photon with an energy smaller than the threshold appears in the final state, are for the idealized detector indistinguishable from the Bhabha scattering events without photon emission. This is the standard textbook approach to the calculation of the real corrections to a given cross-section; admittedly, it is not enough to consider cross-sections measured in realistic experiments, where other aspects must be taken into account (e. g. hard bremsstrahlung effects, detector geometry). Very often, the experimental set up is so complex that the only effective tool to obtain a realistic cross-section is the Monte Carlo method. Nevertheless, we thought it useful to calculate the soft photon emission following this approach, in order to diagrammatically show how the cancellation of the IR divergences works, as well as to provide a benchmark for future, more realistic numerical calculations.

Instead of presenting the lengthy analytical expression of the IR finite differential cross-section, we preferred to implement computer codes that evaluate the differential cross-section at order $\alpha^4(N_F = 1)$ for arbitrary values of the beam energy and scattering angle in the center of mass, E and θ , respectively.

We have found that the corrections at order $\alpha^4(N_F = 1)$ are positive and very small with respect to the corrections of order α^3 (which are negative), so that their constructive contribution is strongly suppressed in the energy range of interest at present and future colliders. The relative weight of the corrections of order α^3 and $\alpha^4(N_F = 1)$ does increase in magnitude with the beam energy and, at a given energy, with the scattering angle. In order to check our numerical consistency, we reproduced the results in [2] and [3] relevant for our purposes.

Our routines for the numerical evaluation of the Bhabha scattering cross-section up to corrections of order $\alpha^4(N_F = 1)$, written both in **Mathematica** [17] and in **Fortran77**, can be obtained from the authors [18].

The paper is organized as follows: in Section 2, we discuss the calculation of the soft real corrections to the Bhabha scattering differential cross-section ¹, up to and including terms of order $\alpha^4(N_F = 1)$. In addition, we investigate the cancellation of the IR divergent terms of the Bhabha scattering virtual cross-section calculated in [11] against the IR divergent terms originating from the soft photon emission diagrams considered here. In Section 3, we present the numerical results obtained

¹Here and in the following, we always consider differential cross-sections summed over the spins of the final state particles and averaged over the spin of the incoming electron and positron.

by evaluating the IR finite Bhabha scattering cross-section (given by the sum of virtual and soft corrections up to terms of order $\alpha^4(N_F = 1)$), for different significant choices of the beam energy E . Also, we compare the complete cross-section with its expansion in powers of the electron mass. We find that the first term in the expansion fits to sufficient approximation the numerical value of the complete cross-section for all the beam energies relevant in present and future colliders. In Section 4, we present our conclusions. Finally, two appendices include the expression of the integrals occurring in the evaluation of the soft radiative corrections relevant to our calculation, as well as the leading terms in the expansion in the limit $m \rightarrow 0$ of the Bhabha scattering differential cross-section we studied.

2 The Real Corrections

In this Section, we discuss the calculation of the real corrections due to the emission of a soft photon to the Bhabha scattering unpolarized differential cross-section in pure QED. In particular, we obtain the real corrections of order α^3 and $\alpha^4(N_F = 1)$. In both cases, we must consider events involving a single soft photon in the final state:

$$e^-(p_1) + e^+(p_2) \longrightarrow e^-(p_3) + e^+(p_4) + \gamma(k), \quad (1)$$

where p_1, p_2, p_3, p_4 , and k are the momenta carried by the incoming electron, incoming positron, outgoing electron, outgoing positron, and outgoing soft photon, respectively. All of the particles in the initial and final states are on-shell, so that $p_i^2 = -m^2$ ($i = 1, 4$) and $k^2 = 0$. Therefore, we introduce the quantities s and s' defined as follows:

$$s = -(p_1 + p_2)^2, \quad s' = -(p_3 + p_4)^2. \quad (2)$$

By definition, the soft photon approximation consists of neglecting k in the numerator of the scattering amplitude and of setting $s' = s$ everywhere. In this approximation, the kinematical relations that link the Mandelstam invariants s, t , and u to the beam energy (E) and to the scattering angle in the center of mass frame (θ), are

$$s = -P^2 \equiv -(p_1 + p_2)^2 \approx -(p_3 + p_4)^2 = 4E^2, \quad (3)$$

$$t = -Q^2 \equiv -(p_1 - p_3)^2 \approx -(p_2 - p_4)^2 = -4(E^2 - m^2) \sin^2 \frac{\theta}{2}, \quad (4)$$

$$u = -V^2 \equiv -(p_1 - p_4)^2 \approx -(p_2 - p_3)^2 = -4(E^2 - m^2) \cos^2 \frac{\theta}{2}, \quad (5)$$

with

$$s + t + u = 4m^2. \quad (6)$$

In the following, we often employ the dimensionless variables x, y , and z , related to the Mandelstam invariants s, t , and u , by the relations

$$s = m^2 \frac{(1+x)^2}{x}, \quad x = \frac{\sqrt{s} - \sqrt{s-4m^2}}{\sqrt{s} + \sqrt{s-4m^2}}, \quad (7)$$

$$t = -m^2 \frac{(1-y)^2}{y}, \quad y = \frac{\sqrt{4m^2-t} - \sqrt{-t}}{\sqrt{4m^2-t} + \sqrt{-t}}, \quad (8)$$

$$u = -m^2 \frac{(1-z)^2}{z}, \quad z = \frac{\sqrt{4m^2-u} - \sqrt{-u}}{\sqrt{4m^2-u} + \sqrt{-u}}, \quad (9)$$

$$(10)$$

which are valid in the physical region $s \geq 4m^2$, $t, u \leq 0$.

The complete Bhabha scattering differential cross-section up to order α^4 ($N_F = 1$) can be written as follows:

$$\frac{d\sigma^T(s, t, m^2)}{d\Omega} = \frac{d\sigma_0(s, t, m^2)}{d\Omega} + \left(\frac{\alpha}{\pi}\right) \frac{d\sigma_1^T(s, t, m^2)}{d\Omega} + \left(\frac{\alpha}{\pi}\right)^2 \frac{d\sigma_2^T(s, t, m^2)}{d\Omega}, \quad (11)$$

where $\sigma_0(s, t, m^2)$ is the tree-level (Born) cross-section

$$\begin{aligned} \frac{d\sigma_0(s, t, m^2)}{d\Omega} &= \frac{\alpha^2}{s} \left\{ \frac{1}{s^2} \left[st + \frac{s^2}{2} + (t - 2m^2)^2 \right] + \frac{1}{t^2} \left[st + \frac{t^2}{2} + (s - 2m^2)^2 \right] \right. \\ &\quad \left. + \frac{1}{st} [(s+t)^2 - 4m^4] \right\}, \end{aligned} \quad (12)$$

and $\sigma_i^T(s, t, m^2)$ ($i = 1, 2$) are the sum of the virtual and real corrections at order α^3 and α^4 ($N_F = 1$), respectively.

2.1 Real Corrections at Order α^3

We first consider the order α^3 contribution to the cross-section, which is given by:

$$\left(\frac{\alpha}{\pi}\right) \frac{d\sigma_1^T(s, t, m^2)}{d\Omega} = \left(\frac{\alpha}{\pi}\right) \left[\frac{d\sigma_1^V(s, t, m^2)}{d\Omega} + \frac{d\sigma_1^S(s, t, m^2)}{d\Omega} \right], \quad (13)$$

where the superscripts V and S stand for “virtual” and “soft”.

The one-loop virtual cross-section $d\sigma_1^V/d\Omega$ can be found in Eq. (67) of [11]; we devote the remaining part of this subsection to the calculation of $d\sigma_1^S/d\Omega$.

The diagrams contributing to the real corrections to the Bhabha scattering cross-section at order α^3 are shown in Fig. 1; the real photon can be emitted by any of the incoming or outgoing fermion lines of the s - and t -channel Bhabha scattering tree-level diagrams.

At this stage, it is convenient to introduce the quantity

$$\frac{d\sigma_0^D(s, t, m^2)}{d\Omega} = \frac{d\sigma_0(s, t, m^2)}{d\Omega} + (D-4) \frac{d\sigma_0^{(D-4)}(s, t, m^2)}{d\Omega} + \mathcal{O}\left((D-4)^2\right), \quad (14)$$

where:

$$\begin{aligned} \frac{d\sigma_0^{(D-4)}(s, t, m^2)}{d\Omega} &= \frac{\alpha}{s} \left\{ \frac{1}{s^2} \left[\frac{s^2}{4} \right] + \frac{1}{t^2} \left[\frac{t^2}{4} \right] \right. \\ &\quad \left. + \frac{1}{st} \left[\frac{1}{2}(s+t)^2 - \frac{1}{2}st - m^2(s+t) \right] \right\}. \end{aligned} \quad (15)$$

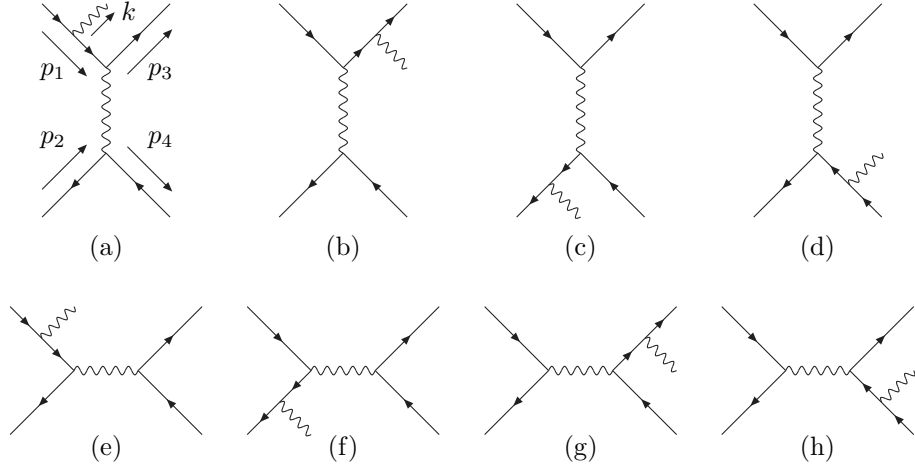


Figure 1: *Diagrams contributing to the real corrections at order α^3 .*

$$\begin{aligned}
& \left[\text{Diagram 1} \right] \times \left[\text{Diagram 2} \right] + 2(J_{11} + J_{13}) \left[\text{Diagram 3} \right] \times \left[\text{Diagram 4} \right] = \text{IR fin.} \\
& \left[\text{Diagram 5} \right] \times \left[\text{Diagram 6} \right] + (J_{11} + J_{13}) \left[\text{Diagram 7} \right] \times \left[\text{Diagram 8} \right] = \text{IR fin.} \\
& \left[\text{Diagram 9} \right] \times \left[\text{Diagram 10} \right] + (J_{11} + J_{12}) \left[\text{Diagram 11} \right] \times \left[\text{Diagram 12} \right] = \text{IR fin.} \\
& \left[\text{Diagram 13} \right] \times \left[\text{Diagram 14} \right] + 2(J_{11} + J_{12}) \left[\text{Diagram 15} \right] \times \left[\text{Diagram 16} \right] = \text{IR fin.}
\end{aligned}$$

Figure 2: *Cancellation of the IR divergencies in the one-loop vertex diagrams.*

$$\begin{aligned}
& \text{Diagram 1} \times \text{Diagram 2} + 4J_{12} \text{Diagram 3} \times \text{Diagram 4} = \text{IR fin.} \\
& \text{Diagram 5} \times \text{Diagram 6} + 2J_{12} \text{Diagram 7} \times \text{Diagram 8} = \text{IR fin.} \\
& \text{Diagram 9} \times \text{Diagram 10} + 4J_{14} \text{Diagram 11} \times \text{Diagram 12} = \text{IR fin.} \\
& \text{Diagram 13} \times \text{Diagram 14} + 2J_{14} \text{Diagram 15} \times \text{Diagram 16} = \text{IR fin.} \\
& \text{Diagram 17} \times \text{Diagram 18} + 2J_{13} \text{Diagram 19} \times \text{Diagram 20} = \text{IR fin.} \\
& \text{Diagram 21} \times \text{Diagram 22} + 4J_{13} \text{Diagram 23} \times \text{Diagram 24} = \text{IR fin.} \\
& \text{Diagram 25} \times \text{Diagram 26} + 2J_{14} \text{Diagram 27} \times \text{Diagram 28} = \text{IR fin.} \\
& \text{Diagram 29} \times \text{Diagram 30} + 4J_{14} \text{Diagram 31} \times \text{Diagram 32} = \text{IR fin.}
\end{aligned}$$

Figure 3: *Cancellation of the IR divergencies in the one-loop box diagrams.*

$d\sigma_0^D/d\Omega$ is the Born Bhabha scattering cross-section obtained by calculating the traces over the Dirac indices in D dimensions. The contribution of the s - and t -channel diagrams and of their interference to the r.h.s. of Eqs. (14,15) is explicit. It is then straightforward to show that, in the soft photon approximation, the contribution of the diagrams in Fig. 1 to the unpolarized differential cross-section is given by

$$\left(\frac{\alpha}{\pi}\right) \frac{d\sigma_1^S(s, t, m^2)}{d\Omega} = \left(\frac{\alpha}{\pi}\right) \frac{d\sigma_0^D(s, t, m^2)}{d\Omega} \sum_{i,j=1}^4 J_{ij}, \quad (16)$$

where the IR divergent quantity J_{ij} is defined as

$$J_{ij} = \epsilon_i \epsilon_j (p_i \cdot p_j) I_{ij}, \quad (17)$$

with $\epsilon_i = +1$ for $i = 1, 4$ and $\epsilon_i = -1$ for $i = 2, 3$, and where I_{ij} indicates the integral:

$$I_{ij} = \frac{1}{\Gamma\left(3 - \frac{D}{2}\right) \pi^{(D-4)/2}} \frac{m^{D-4}}{4\pi^2} \int^\omega \frac{d^D k}{k_0} \frac{1}{(p_i \cdot k)(p_j \cdot k)}. \quad (18)$$

In Eq. (18), D is the dimensional regulator; furthermore the superscript on the integral sign indicates that the integration should be taken over the region $|\vec{k}| = k_0 < \omega$, with ω representing the cut-off on the unobserved soft-photon energy. The integral in Eq. (18) can be evaluated according to the standard technique discussed in detail² in Ref. [16]. It is important to observe that the integrals I_{ij} depend only on the scalar product $p_i \cdot p_j$ (aside from an obvious dependence on E and m), so that

$$\begin{aligned} I_{ij} &= I_{ji}, & I_{11} &= I_{22} = I_{33} = I_{44}, \\ I_{12} &= I_{34}, & I_{13} &= I_{24}, & I_{14} &= I_{23}. \end{aligned} \quad (19)$$

Consequently, the quantities J_{ij} also satisfy the same symmetry relations. Therefore, Eq. (16) becomes

$$\left(\frac{\alpha}{\pi}\right) \frac{d\sigma_1^S(s, t, m^2)}{d\Omega} = \left(\frac{\alpha}{\pi}\right) \frac{d\sigma_0^D(s, t, m^2)}{d\Omega} 4 \sum_{j=1}^4 J_{1j}. \quad (20)$$

The explicit expressions of I_{1j} ($j = 1, 4$) can be found in Appendix A.

²The integral in Eq. (18) and the one in Ref. [16] differ by a normalization factor. In particular, the integral with the normalization employed in this paper can be obtained by multiplying Eq. (1.230) in Ref. [16] by the factor

$$1 - \frac{D-4}{2} \left(\gamma + \ln \pi + \ln \left(\frac{m^2}{\mu^2} \right) \right) + \mathcal{O}((D-4)^2),$$

where γ is the Euler constant and μ the 't Hooft scale. Our choice of the normalization constants removes, in all the divergent integrals, the finite terms associated with the use of dimensional regularization and sets the 't Hooft scale to m ; this choice is consistent with the normalization employed in Ref. [11].

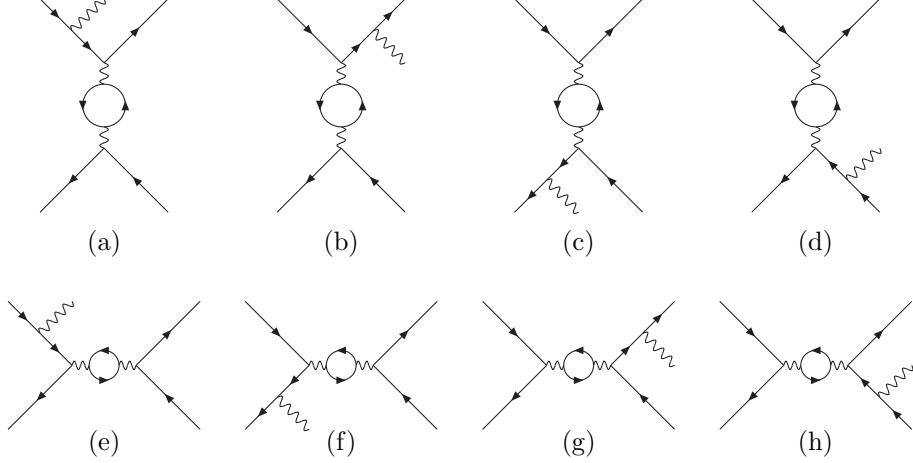


Figure 4: *Diagrams contributing to the real corrections at order $\alpha^4(N_F = 1)$.*

The IR pole that originates from the integral I_{1j} is multiplied, in Eq. (20), by the terms proportional to $(D - 4)$ in σ_0^D ; this product provides a finite contribution to σ_1^S . Terms proportional to $(D - 4)$ in σ_1^S must be neglected.

It is useful to understand how the cancellation of the IR poles works from a diagrammatic point of view. Fig. 2 and Fig. 3 schematically describe the situation. The contribution to the differential cross-section of the interference of the two diagrams shown in the first term of each line is IR divergent. This IR divergence cancels against the contribution to the real cross-section given by the second term in each line, where the product of the two tree-level diagrams represents the contribution of their interference to the cross-section in Born's approximation. We remind the reader that the one-loop photon self-energy diagrams are IR finite.

2.2 Real Corrections at Order $\alpha^4(N_F = 1)$

The order $\alpha^4(N_F = 1)$ contribution to the cross-section is given by:

$$\left(\frac{\alpha}{\pi}\right)^2 \frac{d\sigma_2^T(s, t, m^2)}{d\Omega} = \left(\frac{\alpha}{\pi}\right)^2 \left[\frac{d\sigma_2^V(s, t, m^2)}{d\Omega} + \frac{d\sigma_2^S(s, t, m^2)}{d\Omega} \right], \quad (21)$$

where the two-loop virtual cross-section $d\sigma_2^V/d\Omega$ can be found in Eq. (68) of [11].

The IR divergencies present in the Bhabha scattering differential cross-section at order $\alpha^4(N_F = 1)$ cancel against the contribution to the real cross-section of the interference of the diagrams in Fig. 4 with the single photon emission tree-level diagrams in Fig. 1. In discussing the soft corrections to the cross-section at order $\alpha^4(N_F = 1)$ it is convenient to introduce the quantity

$$\frac{d\sigma_1^D(s, t, m^2)}{d\Omega} \Big|_{(1l, S)} = \frac{d\sigma_1^V(s, t, m^2)}{d\Omega} \Big|_{(1l, S)} + (D - 4) \frac{d\sigma_1^{(D-4)}(s, t, m^2)}{d\Omega} \Big|_{(1l, S)}. \quad (22)$$

The first term in the r.h.s. of the equation above is the contribution to the virtual cross-section of the one-loop self-energy diagrams and corresponding counter-term

diagrams:

$$\begin{aligned} \left. \frac{d\sigma_1^V(s, t, m^2)}{d\Omega} \right|_{(1l, S)} &= \frac{\alpha^2}{s} \left\{ \frac{1}{s^2} \left[st + \frac{s^2}{2} + (t - 2m^2)^2 \right] 2\text{Re}\Pi_0^{(1l, 0)}(s) \right. \\ &\quad + \frac{1}{t^2} \left[st + \frac{t^2}{2} + (s - 2m^2)^2 \right] 2\Pi_0^{(1l, 0)}(t) \\ &\quad \left. + \frac{1}{st} [(s + t)^2 - 4m^4] \left(\text{Re}\Pi_0^{(1l, 0)}(s) + \Pi_0^{(1l, 0)}(t) \right) \right\}, \quad (23) \end{aligned}$$

where $\Pi_0^{(1l, 0)}$ is the UV renormalized photon self-energy. The latter quantity has been discussed in detail in Section 4 of [11]. The second term in the r.h.s. of Eq. (22) is:

$$\begin{aligned} \left. \frac{d\sigma_1^{(D-4)}(s, t, m^2)}{d\Omega} \right|_{(1l, S)} &= \frac{\alpha^2}{s} \left\{ \frac{1}{s^2} \left[st + \frac{s^2}{2} + (t - 2m^2)^2 \right] 2\text{Re}\Pi_0^{(1l, 1)}(s) \right. \\ &\quad + \frac{1}{t^2} \left[st + \frac{t^2}{2} + (s - 2m^2)^2 \right] 2\Pi_0^{(1l, 1)}(t) \\ &\quad + \frac{1}{st} [(s + t)^2 - 4m^4] \left(\text{Re}\Pi_0^{(1l, 1)}(s) + \Pi_0^{(1l, 1)}(t) \right) \\ &\quad + \frac{\text{Re}\Pi_0^{(1l, 0)}(s) + \Pi_0^{(1l, 0)}(t)}{2} + \frac{1}{2} [(s + t)^2 - st - 2m^2(s + t)] \times \\ &\quad \left. \times \left(\text{Re}\Pi_0^{(1l, 0)}(s) + \Pi_0^{(1l, 0)}(t) \right) \right\}, \quad (24) \end{aligned}$$

where $\Pi_0^{(1l, 1)}$ is the term proportional to $(D - 4)$ in the expansion of the renormalized photon self-energy (the explicit expression of this quantity can be found in the appendix of [11]).

The contribution of the soft photon emission to the real corrections to the Bhabha scattering cross-section at order $\alpha^4(N_F = 1)$ is given by

$$\left(\frac{\alpha}{\pi} \right)^2 \frac{d\sigma_2^S(s, t, m^2)}{d\Omega} = \left(\frac{\alpha}{\pi} \right)^2 \left(\left. \frac{d\sigma_1^D(s, t, m^2)}{d\Omega} \right|_{(1l, S)} \right) 4 \sum_{j=1}^4 J_{1j}, \quad (25)$$

where the integrals J_{1j} ($j = 1, \dots, 4$) have been introduced in the previous subsection. The term proportional to $(D - 4)$ in Eq. (22) provides a finite contribution to the real corrections in Eq. (25), since J_{1j} contains an IR pole. Terms proportional to $(D - 4)$ in $\sigma_2^S(s, t, m^2)$ are then neglected.

Figs. 5-8 explicitly show how the cancellation of the IR divergences takes place from a diagrammatic point of view: the contribution to the virtual cross-section of the interference of the diagrams in the first term of each line is IR divergent; such divergence cancels against the interference of the two diagrams in the second term multiplied by the appropriate combination of J_{1j} integrals. We observe that in the last two lines of Fig. 5, in the last line of Fig. 6, and in the second line of Fig. 7, the subtraction of the real radiation in the second term of the l.h.s. does not cancel

$$\begin{aligned}
& \text{Diagram 1} \times \text{Diagram 2} + 2(J_{13} + J_{11}) \text{Diagram 3} \times \text{Diagram 4} = \text{IR fin.} \\
& \text{Diagram 5} \times \text{Diagram 6} + (J_{13} + J_{11}) \text{Diagram 7} \times \text{Diagram 8} = \text{IR fin.} \\
& \text{Diagram 9} \times \text{Diagram 10} + (J_{12} + J_{11}) \text{Diagram 11} \times \text{Diagram 12} = \text{IR pole } \propto \zeta(2) \\
& \text{Diagram 13} \times \text{Diagram 14} + 2(J_{12} + J_{11}) \text{Diagram 15} \times \text{Diagram 16} = \text{IR pole } \propto \zeta(2)
\end{aligned}$$

Figure 5: *Cancellation of the IR divergencies of the two-loop $N_F = 1$ reducible diagrams.*

$$\begin{aligned}
& \text{Diagram 17} \times \text{Diagram 18} + (J_{13} + J_{11}) \text{Diagram 19} \times \text{Diagram 20} = \text{IR fin.} \\
& \text{Diagram 21} \times \text{Diagram 22} + (J_{13} + J_{11}) \text{Diagram 23} \times \text{Diagram 24} = \text{IR fin.} \\
& \text{Diagram 25} \times \text{Diagram 26} + (J_{12} + J_{11}) \text{Diagram 27} \times \text{Diagram 28} = \text{IR fin.} \\
& \text{Diagram 29} \times \text{Diagram 30} + (J_{12} + J_{11}) \text{Diagram 31} \times \text{Diagram 32} = \text{IR pole } \propto \zeta(2)
\end{aligned}$$

Figure 6: *Cancellation of the IR divergencies of the products of one-loop self-energy and vertex diagrams.*

$$\begin{array}{l}
\begin{array}{c} \text{Diagram 1} \end{array} \times \begin{array}{c} \text{Diagram 2} \end{array} + 2J_{12} \begin{array}{c} \text{Diagram 3} \end{array} \times \begin{array}{c} \text{Diagram 4} \end{array} = \text{IR fin.} \\
\begin{array}{c} \text{Diagram 5} \end{array} \times \begin{array}{c} \text{Diagram 6} \end{array} + 2J_{12} \begin{array}{c} \text{Diagram 7} \end{array} \times \begin{array}{c} \text{Diagram 8} \end{array} = \text{IR Pole} \propto \zeta(2) \\
\begin{array}{c} \text{Diagram 9} \end{array} \times \begin{array}{c} \text{Diagram 10} \end{array} + 2J_{14} \begin{array}{c} \text{Diagram 11} \end{array} \times \begin{array}{c} \text{Diagram 12} \end{array} = \text{IR fin.} \\
\begin{array}{c} \text{Diagram 13} \end{array} \times \begin{array}{c} \text{Diagram 14} \end{array} + 2J_{14} \begin{array}{c} \text{Diagram 15} \end{array} \times \begin{array}{c} \text{Diagram 16} \end{array} = \text{IR fin.} \\
\begin{array}{c} \text{Diagram 17} \end{array} \times \begin{array}{c} \text{Diagram 18} \end{array} + 2J_{13} \begin{array}{c} \text{Diagram 19} \end{array} \times \begin{array}{c} \text{Diagram 20} \end{array} = \text{IR fin.} \\
\begin{array}{c} \text{Diagram 21} \end{array} \times \begin{array}{c} \text{Diagram 22} \end{array} + 2J_{13} \begin{array}{c} \text{Diagram 23} \end{array} \times \begin{array}{c} \text{Diagram 24} \end{array} = \text{IR fin.} \\
\begin{array}{c} \text{Diagram 25} \end{array} \times \begin{array}{c} \text{Diagram 26} \end{array} + 2J_{14} \begin{array}{c} \text{Diagram 27} \end{array} \times \begin{array}{c} \text{Diagram 28} \end{array} = \text{IR fin.} \\
\begin{array}{c} \text{Diagram 29} \end{array} \times \begin{array}{c} \text{Diagram 30} \end{array} + 2J_{14} \begin{array}{c} \text{Diagram 31} \end{array} \times \begin{array}{c} \text{Diagram 32} \end{array} = \text{IR fin.}
\end{array}$$

Figure 7: *Cancellation of the IR divergencies in the product of one-loop box and self-energy diagrams.*

Figure 8 displays eight rows of Feynman diagrams illustrating the cancellation of infrared (IR) divergencies in two-loop box diagrams. Each row shows a product of two diagrams, followed by a plus sign, then a coefficient (J₁₂, J₁₃, or J₁₄) multiplied by another product of two diagrams, and finally an equals sign followed by "IR fin." (infrared finite).

- Row 1: $\text{Diagram 1} \times \text{Diagram 2} + J_{12} \text{Diagram 3} \times \text{Diagram 4} = \text{IR fin.}$
- Row 2: $\text{Diagram 5} \times \text{Diagram 6} + J_{12} \text{Diagram 7} \times \text{Diagram 8} = \text{IR fin.}$
- Row 3: $\text{Diagram 9} \times \text{Diagram 10} + J_{14} \text{Diagram 11} \times \text{Diagram 12} = \text{IR fin.}$
- Row 4: $\text{Diagram 13} \times \text{Diagram 14} + J_{14} \text{Diagram 15} \times \text{Diagram 16} = \text{IR fin.}$
- Row 5: $\text{Diagram 17} \times \text{Diagram 18} + J_{13} \text{Diagram 19} \times \text{Diagram 20} = \text{IR fin.}$
- Row 6: $\text{Diagram 21} \times \text{Diagram 22} + J_{13} \text{Diagram 23} \times \text{Diagram 24} = \text{IR fin.}$
- Row 7: $\text{Diagram 25} \times \text{Diagram 26} + J_{14} \text{Diagram 27} \times \text{Diagram 28} = \text{IR fin.}$
- Row 8: $\text{Diagram 29} \times \text{Diagram 30} + J_{14} \text{Diagram 31} \times \text{Diagram 32} = \text{IR fin.}$

Figure 8: *Cancellation of the IR divergencies in the two-loop box diagrams.*

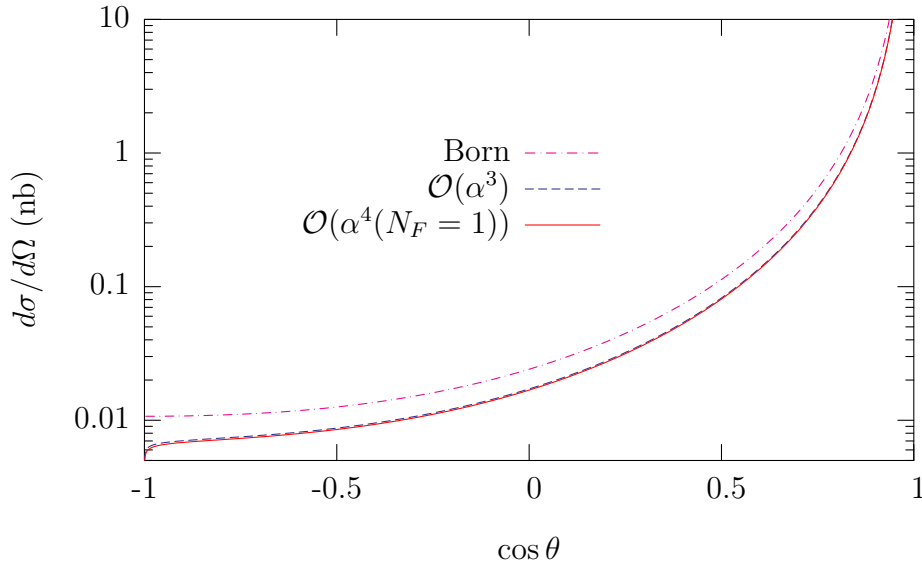


Figure 9: *Differential cross-section for $E = 22$ GeV. The dashed-dotted line corresponds to the Born cross-section, the dashed one to the cross-section up to corrections of order α^3 , and the continuous one to the cross-section up to and including corrections of order $\alpha^4(N_F = 1)$.*

completely the IR pole in the corresponding virtual correction (first term in the l.h.s.). A residual IR pole, proportional to $\zeta(2)$, remains. As expected, the sum of the residual poles vanishes and the cross-section is therefore IR finite.

3 Numerical Results

In order to numerically evaluate the Bhabha scattering cross-section up to corrections of order $\alpha^4(N_F = 1)$, we developed two computer codes. One of them was written in **Mathematica** [17], while the other was written in **Fortran77**. Following [3], in the numerical calculation, we fixed the energy cut-off on the undetected soft photon to $\omega = 0.1 E$, where E is the beam energy. We compared the results of the two codes for a beam energy ranging from $E = 0.01$ GeV to $E = 500$ GeV, and for an arbitrary choice of the scattering angle θ , finding complete agreement.

In Fig. 9, we show the differential cross-section for $E = 22$ GeV and for $0 < \theta < \pi$. The dashed-dotted line represents the cross-section in the Born approximation (Eq. (12)), while the continuous (dashed) line represents the cross-section at order α^3 ($\alpha^4(N_F = 1)$), Eqs. (13,21). The radiative corrections at order α^3 are negative; therefore, they lower the cross-section. The corrections at order $\alpha^4(N_F = 1)$ are negative and very small with respect to the corrections of order α^3 ; they lower the differential cross-section, although, as seen in Fig. 9, the effect is difficult to appreciate graphically.

The relative weight of the corrections of order α^3 and $\alpha^4(N_F = 1)$ are shown

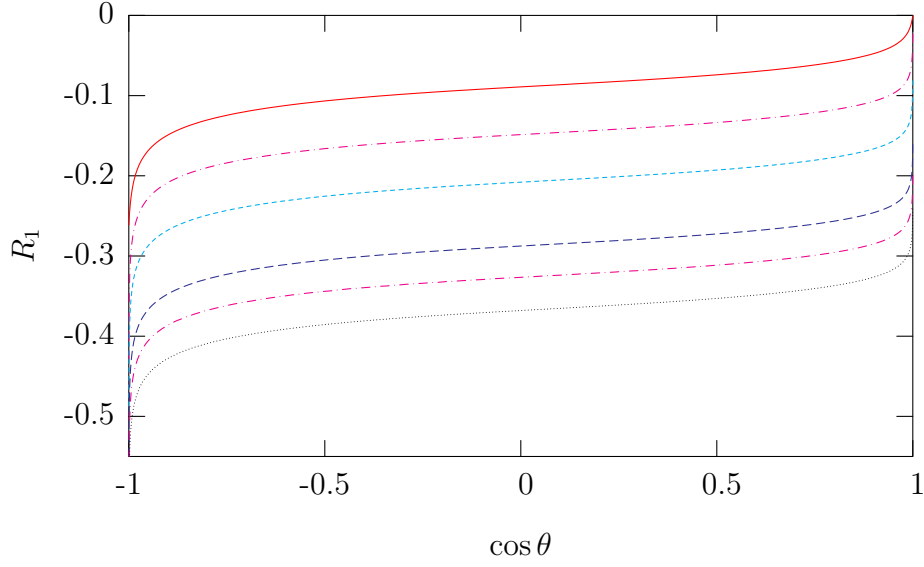


Figure 10: *Ratio R_1 of the one-loop corrections over the tree-level cross-section. The different lines correspond to the following beam energies (from top to bottom): $E = 10$ MeV, 100 MeV, 1 GeV, 22 GeV, 100 GeV, and 500 GeV.*

in Fig. 10 and Fig. 11 for six different choices of the beam energy. In Fig. 10, we plotted the ratio of the order α^3 corrections to the tree-level cross-section:

$$R_1 = \frac{\alpha}{\pi} \left(\frac{d\sigma_1^T}{d\Omega} \right) \left(\frac{d\sigma_0}{d\Omega} \right)^{-1}. \quad (26)$$

It is evident that the relative weight of the correction increases in magnitude with the beam energy and, at a given energy, with the scattering angle. A similar plot can be found in [3], for $E = 22$ GeV, where the full set of one-loop corrections in the Standard Model were considered. At $\theta = \pi/2$ the corrections range from -9% of the Born cross-section at $E = 10$ MeV to -37% for $E = 500$ GeV.

In Fig. 11, we plotted the ratio of the order $\alpha^4(N_F = 1)$ corrections to the complete order α^3 cross-section:

$$R_2 = \left(\frac{\alpha}{\pi} \right)^2 \left(\frac{d\sigma_2^T}{d\Omega} \right) \left(\frac{d\sigma_0}{d\Omega} + \frac{\alpha}{\pi} \frac{d\sigma_1^T}{d\Omega} \right)^{-1}. \quad (27)$$

The corrections increase in magnitude with the energy and, for fixed energy, with the scattering angle. The relative weight of these corrections at $\theta = \pi/2$ ranges from -0.08% of the complete cross-section at order α^3 for $E = 10$ MeV to -4.3% for $E = 500$ GeV.

Figs. 12 and 13 show the dependence of the cross-section on the beam energy, for small and large scattering angles respectively. Similar plots for a wider choices of angles can be found, limited to the one-loop corrections, in [2]. Using our codes, we reproduced the plots shown in [2] finding agreement.

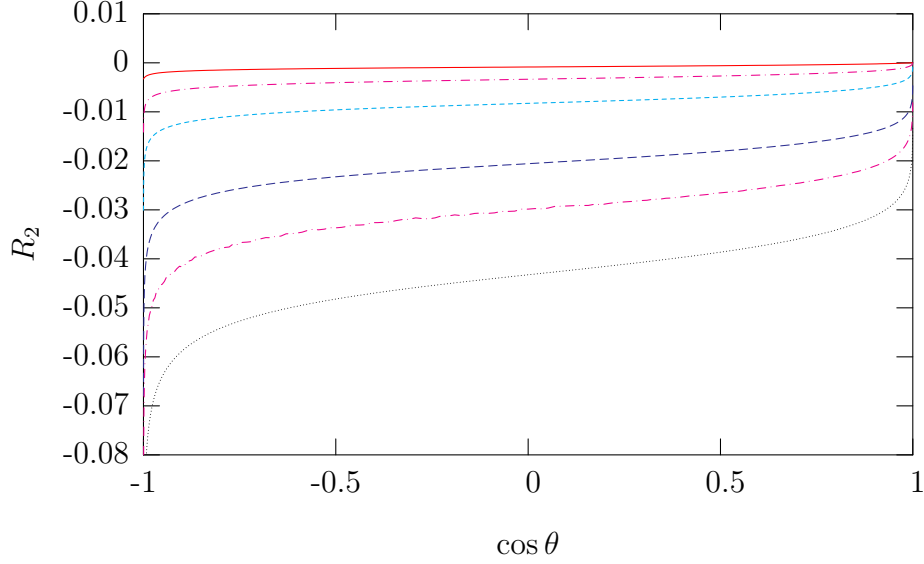


Figure 11: Ratio R_2 of the two-loop corrections over the tree-level cross-section. The different lines correspond to the following beam energies (from top to bottom): $E = 10$ MeV, 100 MeV, 1 GeV, 22 GeV, 100 GeV, and 500 GeV.

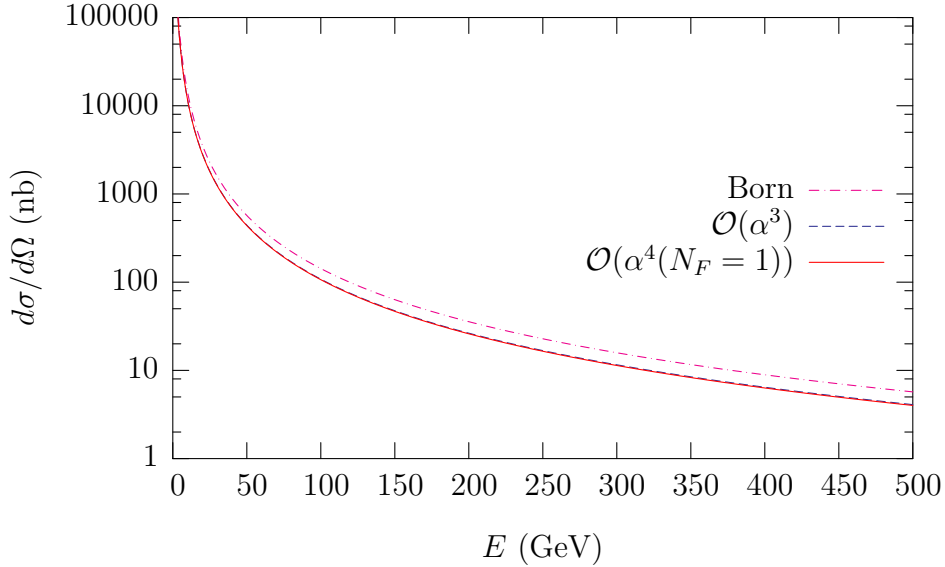


Figure 12: Bhabha scattering cross-section as a function of the beam energy for a scattering angle $\theta = 5^\circ$. The dashed-dotted line corresponds to the Born cross-section, the dashed one to the cross-section up to corrections of order α^3 , and the continuous one to the cross-section up to corrections of order $\alpha^4(N_F = 1)$.

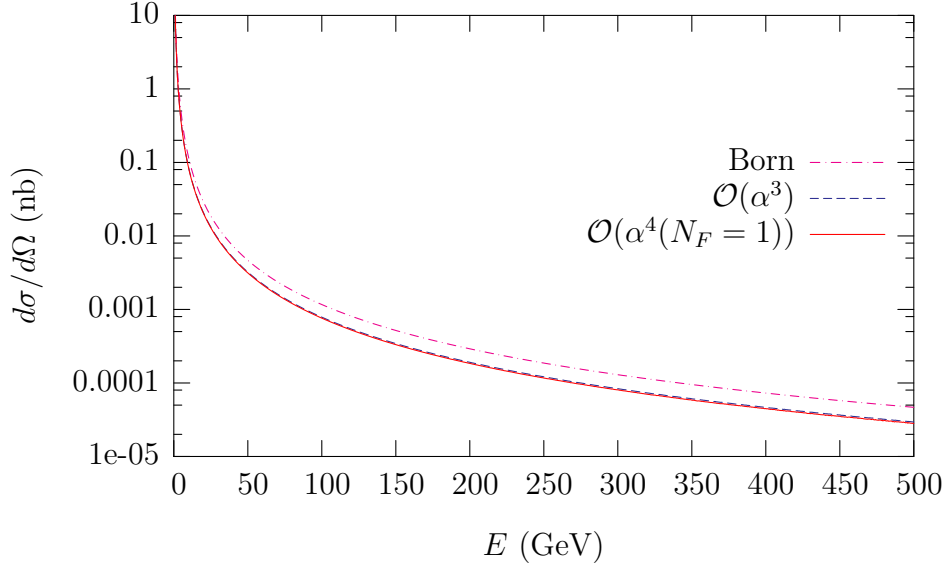


Figure 13: *Bhabha scattering cross-section as a function of the beam energy for a scattering angle $\theta = 90^\circ$. The dashed-dotted line corresponds to the Born cross-section, the dashed one to the cross-section up to corrections of order α^3 , and the continuous one to the cross-section up to corrections of order $\alpha^4(N_F = 1)$.*

To complete the analysis of the numerical results, we expanded the analytic expression of the cross-section in the limit in which the squared electron mass is negligible with respect to the kinematic invariants s , t , and u . We define the leading terms of the cross-section in the limit $m^2 \rightarrow 0$ through the relations:

$$\frac{d\sigma_i^T}{d\Omega} = \left. \frac{d\sigma_i^T}{d\Omega} \right|_L + \mathcal{O}\left(\frac{m^2}{s}, \frac{m^2}{t}, \frac{m^2}{u}\right), \quad (28)$$

where $i = 1, 2$ and where the subscript “ L ” stands for “leading”. The expressions of $(d\sigma_i^T/d\Omega)|_L$ can be found in Appendix B.

In Figs. 14 and 15 we plotted (for a fixed value of the scattering angle) the quantities:

$$D_1 = \frac{\alpha}{\pi} \left(\left. \frac{d\sigma_1^T}{d\Omega} \right|_L - \frac{d\sigma_1^T}{d\Omega} \right) \left(\frac{d\sigma_0}{d\Omega} \right)^{-1}, \quad (29)$$

$$D_2 = \left(\frac{\alpha}{\pi} \right)^2 \left(\left. \frac{d\sigma_2^T}{d\Omega} \right|_L - \frac{d\sigma_2^T}{d\Omega} \right) \left(\frac{d\sigma_0}{d\Omega} + \frac{\alpha}{\pi} \frac{d\sigma_1^T}{d\Omega} \right)^{-1}, \quad (30)$$

respectively. A glance of the figures shows that the leading terms of the cross-section approximate to a continuously better degree the complete results for increasing beam energy. The leading terms of the cross-section fail to reproduce the complete result in the extremely forward and backward regions, where t or u becomes smaller than m^2 .

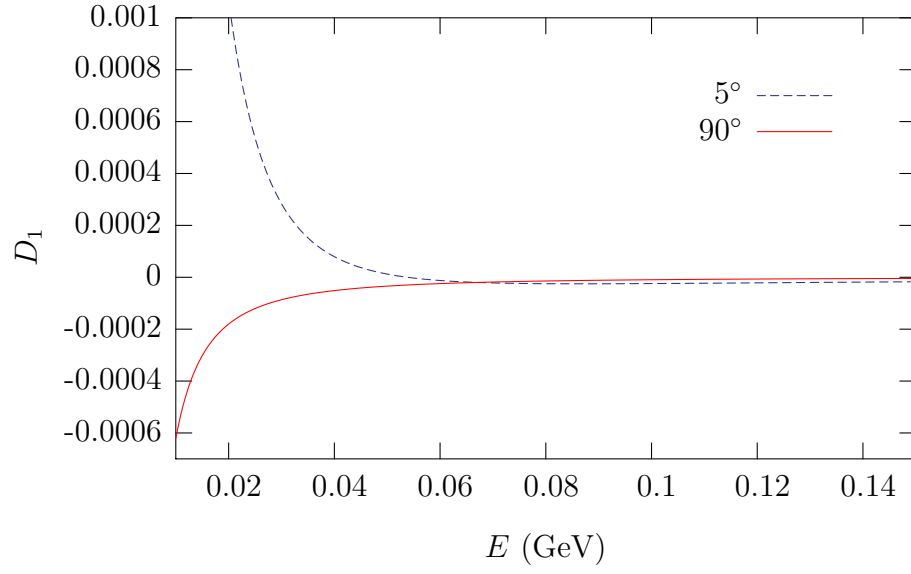


Figure 14: D_1 for $\theta = 5^\circ$ (dashed line) and 90° (continuous line).

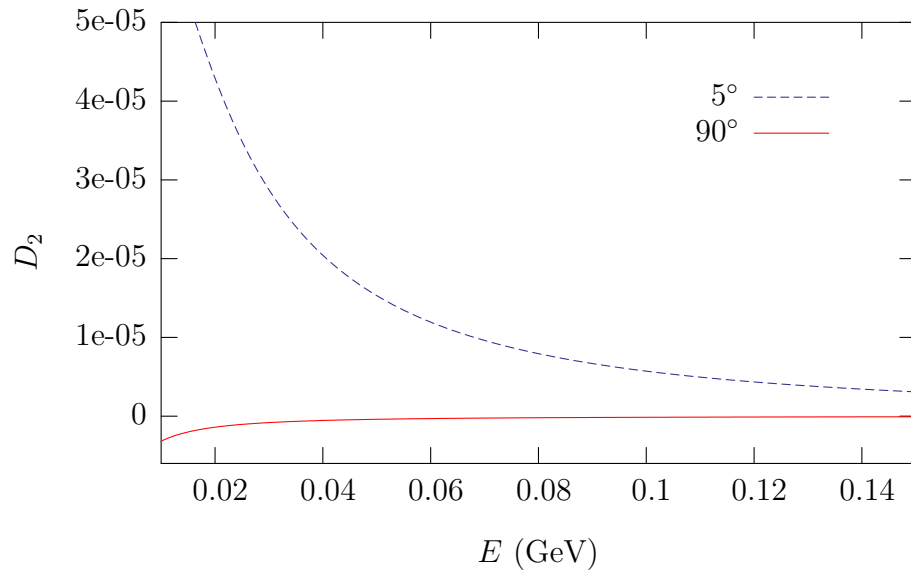


Figure 15: D_2 for $\theta = 5^\circ$ (dashed line) and 90° (continuous line).

The codes for the numerical evaluation of the Bhabha scattering differential cross-section in pure QED up to order $\alpha^4(N_F = 1)$ are available from the authors [18].

4 Conclusions

In the present paper, we completed the evaluation of the Bhabha scattering cross-section at order $\alpha^4(N_F = 1)$ in pure QED. The calculation was performed without neglecting the electron mass m , and is valid for all physical values of the independent Mandelstam invariants s and t . The master integrals necessary for the evaluation of the virtual corrections were calculated in [8] and [9], while the UV renormalized unpolarized differential cross-section was obtained in [11]. The calculation was completed by providing the real correction in the approximation of a soft photon emission up to order $\alpha^4(N_F = 1)$, as well as by explicitly showing that the IR poles present in such corrections cancel the remaining IR poles of the virtual cross-section calculated in [11]. Finally, we developed computer codes for the numerical evaluation of the UV and IR finite cross-section. We compared our findings with the results present in the literature (where possible), obtaining complete agreement. We verified that the effect of the terms proportional to positive powers of the electron mass m is negligible in the energy range of interest in present and future colliders.

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A Integral I_{ij}

As explained in Section 2, the integrals I_{ij} are symmetric with respect to their indices and satisfy the symmetry properties listed in Eqs. (19). Therefore, in order to calculate the real corrections to the Bhabha scattering cross-section up to the order that we consider in this work, one needs to calculate the integrals I_{1j} ($j = 1, 4$). Following the procedure outlined in Ref. [16], it is straightforward to verify that,

after the integration on the photon phase space has been carried out, the integral becomes

$$I_{1j} = \frac{\rho_{1j}}{2} \left[\left(\frac{1}{D-4} + \frac{1}{2} \ln \left(\frac{\omega^2}{m^2} \right) + \ln 2 \right) I_{1j}^{(0)} - \Delta I_{1j} \right] + \mathcal{O}(D-4). \quad (31)$$

In the equation above, the quantity ρ_{ij} has a particularly simple expression in terms of the dimensionless quantities x , y , and z introduced in Eqs. (7-9):

$$\rho_{12} = \frac{1}{x}, \quad \rho_{13} = \frac{1}{y}, \quad \rho_{14} = \frac{1}{z}, \quad \rho_{11} = 1. \quad (32)$$

The quantity $I_{1j}^{(0)}$ in Eq. (31) can be expressed as a simple integral as follows:

$$I_{1j}^{(0)} = \frac{2}{m^2} \int_0^1 dr \left[1 - 2r \left(1 + \rho_{1j} \frac{p_1 \cdot p_j}{m^2} \right) \right]^{-1}. \quad (33)$$

After observing that the scalar products $p_1 \cdot p_j$ also have a simple form in terms of x , y , and z ,

$$p_1 \cdot p_2 = -\frac{m^2}{2} \frac{1+x^2}{x}, \quad p_1 \cdot p_3 = -\frac{m^2}{2} \frac{1+y^2}{y}, \quad p_1 \cdot p_4 = -\frac{m^2}{2} \frac{1+z^2}{z}, \quad p_1^2 = -m^2, \quad (34)$$

one finds:

$$I_{11}^{(0)} = \frac{2}{m^2}, \quad (35)$$

$$I_{12}^{(0)} = -\frac{4}{m^2} \frac{x^2}{1-x^2} \ln x, \quad (36)$$

$$I_{13}^{(0)} = -\frac{4}{m^2} \frac{y^2}{1-y^2} \ln y, \quad (37)$$

$$I_{14}^{(0)} = -\frac{4}{m^2} \frac{z^2}{1-z^2} \ln z. \quad (38)$$

In addition, the quantity ΔI_{1j} can also be expressed in integral form:

$$\Delta I_{1j} = \int_0^1 dr \frac{1}{(P_{1j})^2} \frac{(P_{1j})_0}{|\vec{P}_{1j}|} \ln \frac{(P_{1j})_0 - |\vec{P}_{1j}|}{(P_{1j})_0 + |\vec{P}_{1j}|}, \quad (39)$$

where the Lorentz vector P_{1j}^μ is defined by the relation

$$P_{1j}^\mu = p_j^\mu + r (\rho_{1j} p_1^\mu - p_j^\mu). \quad (40)$$

The integral in Eq. (39) can also be evaluated according to the procedure outlined in Ref. [16]; one finds:

$$\Delta I_{11} = -\frac{1}{m^2} \frac{1+x}{1-x} \ln x, \quad (41)$$

$$\Delta I_{1l} = -\frac{2}{m^2(\rho_{1l}^2 - 1)} \left[\text{Li}_2(a_l^{(1)}) + \text{Li}_2(a_l^{(2)}) - \text{Li}_2(a_l^{(3)}) - \text{Li}_2(a_l^{(4)}) \right], \quad (42)$$

where $l = 2, 3, 4$ and the arguments $a_l^{(k)}$ ($k = 1, \dots, 4$) have convenient expressions in terms of x and ρ_{1l} :

$$a_l^{(1)} = \frac{1 - x\rho_{1l}}{1 + \rho_{1l}}, \quad a_l^{(2)} = \frac{x - \rho_{1l}}{x(1 + \rho_{1l})}, \quad (43)$$

$$a_l^{(3)} = \frac{\rho_{1l} - x}{1 + \rho_{1l}}, \quad a_l^{(4)} = -\frac{1 - x\rho_{1l}}{x(1 + \rho_{1l})}. \quad (44)$$

We observe that $a_2^{(1)}$ and $a_2^{(4)}$ are equal to zero; as a consequence, the corresponding dilogarithms in Eq. (42) also vanish.

B Leading Corrections at Order α^3 and Order $\alpha^4(N_F = 1)$

In this Appendix, we provide the explicit expressions of the leading radiative corrections defined in Eq. (28).

$$\begin{aligned} \left. \frac{d\sigma_1^T}{d\Omega} \right|_L = & \alpha^2 \left\{ -\ln\left(\frac{m^2}{s}\right) \left(\frac{11}{3} \frac{t^2}{s^3} + \frac{22}{3} \frac{t}{s^2} + \frac{11}{s} + \frac{11}{3} \frac{s}{t^2} + \frac{22}{3t} \right) - \ln^2\left(-\frac{t}{s}\right) \left(\frac{t^2}{s^3} + \frac{13}{4} \frac{t}{s^2} \right. \right. \\ & + \frac{25}{4s} + 2 \frac{s}{t^2} + \frac{19}{4t} \Big) + \ln\left(-\frac{t}{s}\right) \ln\left(-\frac{u}{s}\right) \left(2 \frac{t^2}{s^3} + 5 \frac{t}{s^2} + \frac{19}{2s} + 4 \frac{s}{t^2} + \frac{8}{t} \right) \\ & + \ln\left(-\frac{t}{s}\right) \left(\frac{11}{6} \frac{t}{s^2} + \frac{11}{2s} + \frac{11}{3} \frac{s}{t^2} + \frac{5}{t} \right) - \ln^2\left(-\frac{u}{s}\right) \left(\frac{t^2}{s^3} + \frac{5}{2} \frac{t}{s^2} + \frac{7}{2s} + \frac{5}{2t} + \frac{s}{t^2} \right) \\ & + \ln\left(-\frac{u}{s}\right) \left(\frac{t}{2s^2} + \frac{1}{2t} \right) + \zeta(2) \left(2 \frac{t^2}{s^3} + \frac{t}{s^2} - \frac{9}{2s} - 4 \frac{s}{t^2} - \frac{8}{t} \right) - \frac{46}{9} \frac{t^2}{s^3} - \frac{92}{9} \frac{t}{s^2} \\ & - \frac{46}{3s} - \frac{46}{9} \frac{s}{t^2} - \frac{92}{9t} - \left(\frac{t^2}{s^3} + 2 \frac{t}{s^2} + \frac{3}{s} + \frac{s}{t^2} + \frac{2}{t} \right) \left[4 \ln(2) \left(1 + \ln\left(\frac{m^2}{s}\right) \right. \right. \\ & + \ln\left(-\frac{u}{s}\right) - \ln\left(-\frac{t}{s}\right) \Big) + 4 \text{Li}_2\left(-\frac{u}{t}\right) + 2 \ln\left(\frac{\omega^2}{s}\right) \left(1 + \ln\left(\frac{m^2}{s}\right) \right. \\ & \left. \left. - \ln\left(-\frac{t}{s}\right) + \ln\left(-\frac{u}{s}\right) \right) \right] \Big\}. \quad (45) \end{aligned}$$

$$\begin{aligned} \left. \frac{d\sigma_2^T}{d\Omega} \right|_L = & -\alpha^2 \left\{ \zeta(2) \left(\frac{86}{9} \frac{t^2}{s^3} + \frac{263}{18} \frac{t}{s^2} + \frac{31}{3s} + \frac{31}{9} \frac{s}{t^2} + \frac{61}{18t} \right) - \zeta(3) \left(2 \frac{t^2}{s^3} + 4 \frac{t}{s^2} + \frac{6}{s} \right. \right. \\ & + 2 \frac{s}{t^2} + \frac{4}{t} \Big) + \text{Li}_2\left(-\frac{t}{s}\right) \ln\left(-\frac{t}{s}\right) \left(\frac{2}{3} \frac{t^2}{s^3} + \frac{4}{3} \frac{t}{s^2} + \frac{2}{s} + \frac{2}{3} \frac{s}{t^2} + \frac{4}{3t} \right) \\ & - \text{Li}_2\left(-\frac{u}{s}\right) \ln\left(-\frac{u}{s}\right) \left(\frac{2}{3} \frac{t^2}{s^3} + \frac{t}{s^2} + \frac{1}{s} + \frac{1}{3t} \right) - \text{Li}_2\left(-\frac{u}{t}\right) \ln\left(\frac{m^2}{s}\right) \times \\ & \times \left(\frac{8}{3} \frac{t^2}{s^3} + \frac{16}{3} \frac{t}{s^2} + \frac{8}{s} + \frac{8}{3} \frac{s}{t^2} + \frac{16}{3t} \right) + \text{Li}_2\left(-\frac{u}{t}\right) \ln\left(-\frac{t}{s}\right) \left(\frac{5}{3} \frac{t}{s^2} + \frac{5}{s} \right. \end{aligned}$$

$$\begin{aligned}
& + \frac{10}{3} \frac{s}{t^2} + \frac{5}{t} \Big) - \text{Li}_2 \left(-\frac{u}{t} \right) \ln \left(-\frac{u}{s} \right) \left(\frac{1}{3} \frac{t}{s^2} + \frac{1}{s} + \frac{2}{3} \frac{s}{t^2} + \frac{1}{t} \right) \\
& - \text{Li}_2 \left(-\frac{u}{t} \right) \left(\frac{40}{9} \frac{t^2}{s^3} + \frac{80}{9} \frac{t}{s^2} + \frac{40}{3s} + \frac{40}{9} \frac{s}{t^2} + \frac{80}{9t} \right) - \text{Li}_3 \left(-\frac{t}{s} \right) \left(\frac{2}{3} \frac{t^2}{s^3} \right. \\
& + \frac{4}{3} \frac{t}{s^2} + \frac{2}{s} + \frac{2}{3} \frac{s}{t^2} + \frac{4}{3t} \Big) + \text{Li}_3 \left(-\frac{u}{s} \right) \left(\frac{2}{3} \frac{t^2}{s^3} + \frac{t}{s^2} + \frac{1}{s} + \frac{1}{3t} \right) \\
& + \text{Li}_3 \left(-\frac{t}{u} \right) \left(\frac{1}{3} \frac{t}{s^2} + \frac{1}{s} + \frac{2}{3} \frac{s}{t^2} + \frac{1}{t} \right) + \ln \left(\frac{m^2}{s} \right) \zeta(2) \left(\frac{8}{3} \frac{t^2}{s^3} + \frac{7}{3} \frac{t}{s^2} - \frac{5}{2s} \right. \\
& - \frac{10}{3} \frac{s}{t^2} - \frac{20}{3t} \Big) - \ln^3 \left(\frac{m^2}{s} \right) \left(\frac{1}{9} \frac{t^2}{s^3} + \frac{2}{9} \frac{t}{s^2} + \frac{1}{3s} + \frac{1}{9} \frac{s}{t^2} + \frac{2}{9t} \right) + \ln^2 \left(\frac{m^2}{s} \right) \times \\
& \times \ln \left(-\frac{t}{s} \right) \left(\frac{1}{3} \frac{t^2}{s^3} + \frac{2}{3} \frac{t}{s^2} + \frac{1}{s} + \frac{1}{3} \frac{s}{t^2} + \frac{2}{3t} \right) - \ln^2 \left(\frac{m^2}{s} \right) \ln \left(-\frac{u}{s} \right) \left(\frac{1}{3} \frac{t^2}{s^3} \right. \\
& + \frac{2}{3} \frac{t}{s^2} + \frac{1}{s} + \frac{s}{3t^2} + \frac{2}{3t} \Big) - \ln^2 \left(\frac{m^2}{s} \right) \ln \left(\frac{\omega^2}{s} \right) \left(\frac{4}{3} \frac{t^2}{s^3} + \frac{8}{3} \frac{t}{s^2} + \frac{4}{s} + \frac{4}{3} \frac{s}{t^2} \right. \\
& + \frac{8}{3t} \Big) - \ln^2 \left(\frac{m^2}{s} \right) \left(\frac{55}{18} \frac{t^2}{s^3} + \frac{55}{9} \frac{t}{s^2} + \frac{55}{6s} + \frac{55}{18} \frac{s}{t^2} + \frac{55}{9t} \right) - \ln \left(\frac{m^2}{s} \right) \times \\
& \times \ln^2 \left(-\frac{t}{s} \right) \left(\frac{2}{3} \frac{t^2}{s^3} + \frac{31}{12} \frac{t}{s^2} + \frac{21}{4s} + \frac{5}{3} \frac{s}{t^2} + \frac{49}{12t} \right) + \ln \left(\frac{m^2}{s} \right) \ln \left(-\frac{t}{s} \right) \times \\
& \times \ln \left(-\frac{u}{s} \right) \left(\frac{4}{3} \frac{t^2}{s^3} + \frac{11}{3} \frac{t}{s^2} + \frac{15}{2s} + \frac{10}{3} \frac{s}{t^2} + \frac{20}{3t} \right) + \ln \left(\frac{m^2}{s} \right) \ln \left(-\frac{t}{s} \right) \times \\
& \times \ln \left(\frac{\omega^2}{s} \right) \left(\frac{4}{3} \frac{t^2}{s^3} + \frac{10}{3} \frac{t}{s^2} + \frac{6}{s} + \frac{8}{3} \frac{s}{t^2} + \frac{14}{3t} \right) + \ln \left(\frac{m^2}{s} \right) \ln \left(-\frac{t}{s} \right) \times \\
& \times \left(\frac{10}{9} \frac{t^2}{s^3} + \frac{85}{18} \frac{t}{s^2} + \frac{65}{6s} + \frac{55}{9} \frac{s}{t^2} + \frac{83}{9t} \right) - \ln \left(\frac{m^2}{s} \right) \ln^2 \left(-\frac{u}{s} \right) \left(\frac{2}{3} \frac{t^2}{s^3} + \frac{11}{6} \frac{t}{s^2} \right. \\
& + \frac{5}{2s} + \frac{2}{3} \frac{s}{t^2} + \frac{11}{6t} \Big) - \ln \left(\frac{m^2}{s} \right) \ln \left(-\frac{u}{s} \right) \ln \left(\frac{\omega^2}{s} \right) \left(\frac{4}{3} \frac{t^2}{s^3} + \frac{8}{3} \frac{t}{s^2} + \frac{4}{s} + \frac{4}{3} \frac{s}{t^2} \right. \\
& + \frac{8}{3t} \Big) - \ln \left(\frac{m^2}{s} \right) \ln \left(-\frac{u}{s} \right) \left(\frac{10}{9} \frac{t^2}{s^3} + \frac{31}{18} \frac{t}{s^2} + \frac{10}{3s} + \frac{10}{9} \frac{s}{t^2} + \frac{31}{18t} \right) \\
& - \ln \left(\frac{m^2}{s} \right) \ln \left(\frac{\omega^2}{s} \right) \left(\frac{32}{9} \frac{t^2}{s^3} + \frac{64}{9} \frac{t}{s^2} + \frac{32}{3s} + \frac{32}{9} \frac{s}{t^2} + \frac{64}{9t} \right) \\
& - \ln \left(\frac{m^2}{s} \right) \left(\frac{281}{27} \frac{t^2}{s^3} + \frac{562}{27} \frac{t}{s^2} + \frac{281}{9s} + \frac{281}{27} \frac{s}{t^2} + \frac{562}{27t} \right) - \ln \left(-\frac{t}{s} \right) \zeta(2) \left(2 \frac{t^2}{s^3} \right. \\
& + \frac{5}{3} \frac{t}{s^2} - \frac{31}{6s} - 4 \frac{s}{t^2} - \frac{9}{t} \Big) - \ln^3 \left(-\frac{t}{s} \right) \left(\frac{1}{9} \frac{t^2}{s^3} - \frac{7}{9} \frac{t}{s^2} - \frac{28}{9s} - \frac{16}{9} \frac{s}{t^2} - \frac{121}{36t} \right) \\
& + \ln^2 \left(-\frac{t}{s} \right) \ln \left(-\frac{u}{s} \right) \left(\frac{1}{3} \frac{t^2}{s^3} - \frac{5}{6} \frac{t}{s^2} - \frac{25}{6s} - \frac{10}{3} \frac{s}{t^2} - \frac{31}{6t} \right) - \ln^2 \left(-\frac{t}{s} \right) \times \\
& \times \ln \left(\frac{\omega^2}{s} \right) \left(\frac{2}{3} \frac{t}{s^2} + \frac{2}{s} + \frac{4}{3} \frac{s}{t^2} + \frac{2}{t} \right) - \ln^2 \left(-\frac{t}{s} \right) \left(\frac{10}{9} \frac{t^2}{s^3} + \frac{44}{9} \frac{t}{s^2} + \frac{34}{3s} + \frac{95}{18} \frac{s}{t^2} \right. \\
& + \frac{86}{9t} \Big) - \ln \left(-\frac{t}{s} \right) \ln^2 \left(-\frac{u}{s} \right) \left(\frac{1}{3} \frac{t^2}{s^3} - \frac{13}{12s} - \frac{s}{t^2} - \frac{3}{2t} \right) + \ln \left(-\frac{t}{s} \right) \times
\end{aligned}$$

$$\begin{aligned}
& \times \ln\left(-\frac{u}{s}\right) \ln\left(\frac{\omega^2}{s}\right) \left(\frac{2}{3} \frac{t}{s^2} + \frac{2}{s} + \frac{4}{3} \frac{s}{t^2} + \frac{2}{t}\right) + \ln\left(-\frac{t}{s}\right) \ln\left(-\frac{u}{s}\right) \left(\frac{20}{9} \frac{t^2}{s^3} \right. \\
& + \frac{55}{9} \frac{t}{s^2} + \frac{77}{6s} + \frac{50}{9} \frac{s}{t^2} + \frac{197}{18t}\bigg) + \ln\left(-\frac{t}{s}\right) \ln\left(\frac{\omega^2}{s}\right) \left(\frac{20}{9} \frac{t^2}{s^3} + \frac{46}{9} \frac{t}{s^2} + \frac{26}{3s} \right. \\
& + \frac{32}{9} \frac{s}{t^2} + \frac{58}{9t}\bigg) + \ln\left(-\frac{t}{s}\right) \left(\frac{56}{27} \frac{t^2}{s^3} + \frac{449}{54} \frac{t}{s^2} + \frac{337}{18s} + \frac{281}{27} \frac{s}{t^2} + \frac{418}{27t}\right) \\
& + \ln\left(-\frac{u}{s}\right) \zeta(2) \left(2 \frac{t^2}{s^3} + \frac{19}{3} \frac{t}{s^2} + \frac{29}{3s} + \frac{4}{3} \frac{s}{t^2} + \frac{17}{3t}\right) - \ln^2\left(-\frac{u}{s}\right) \left(\frac{10}{9} \frac{t^2}{s^3} \right. \\
& + \frac{29}{9} \frac{t}{s^2} + \frac{9}{2s} + \frac{10}{9} \frac{s}{t^2} + \frac{29}{9t}\bigg) + \ln^3\left(-\frac{u}{s}\right) \left(\frac{1}{9} \frac{t^2}{s^3} + \frac{5}{18} \frac{t}{s^2} + \frac{5}{18s} + \frac{1}{6t}\right) \\
& - \ln\left(-\frac{u}{s}\right) \ln\left(\frac{\omega^2}{s}\right) \left(\frac{20}{9} \frac{t^2}{s^3} + \frac{40}{9} \frac{t}{s^2} + \frac{20}{3s} + \frac{20}{9} \frac{s}{t^2} + \frac{40}{9t}\right) \\
& - \ln\left(-\frac{u}{s}\right) \left(\frac{56}{27} \frac{t^2}{s^3} + \frac{161}{54} \frac{t}{s^2} + \frac{56}{9s} + \frac{56}{27} \frac{s}{t^2} + \frac{161}{54t}\right) - \ln\left(\frac{\omega^2}{s}\right) \left(\frac{20}{9} \frac{t^2}{s^3} \right. \\
& + \frac{40}{9} \frac{t}{s^2} + \frac{20}{3s} + \frac{20}{9} \frac{s}{t^2} + \frac{40}{9t}\bigg) - \ln(2) \ln^2\left(\frac{m^2}{s}\right) \left(\frac{8}{3} \frac{t^2}{s^3} + \frac{16}{3} \frac{t}{s^2} + \frac{8}{s} \right. \\
& + \frac{8}{3} \frac{s}{t^2} + \frac{16}{3t}\bigg) + \ln(2) \ln\left(\frac{m^2}{s}\right) \ln\left(-\frac{t}{s}\right) \left(\frac{8}{3} \frac{t^2}{s^3} + \frac{20}{3} \frac{t}{s^2} + \frac{12}{s} + \frac{16}{3} \frac{s}{t^2} \right. \\
& + \frac{28}{3t}\bigg) - \ln(2) \ln\left(\frac{m^2}{s}\right) \ln\left(-\frac{u}{s}\right) \left(\frac{8}{3} \frac{t^2}{s^3} + \frac{16}{3} \frac{t}{s^2} + \frac{8}{s} + \frac{8}{3} \frac{s}{t^2} + \frac{16}{3t}\right) \\
& - \ln(2) \ln\left(\frac{m^2}{s}\right) \left(\frac{64}{9} \frac{t^2}{s^3} + \frac{128}{9} \frac{t}{s^2} + \frac{64}{3s} + \frac{64}{9} \frac{s}{t^2} + \frac{128}{9t}\right) \\
& - \ln(2) \ln^2\left(-\frac{t}{s}\right) \left(\frac{4}{3} \frac{t}{s^2} + \frac{4}{s} + \frac{8}{3} \frac{s}{t^2} + \frac{4}{t}\right) + \ln(2) \ln\left(-\frac{t}{s}\right) \ln\left(-\frac{u}{s}\right) \left(\frac{4}{3} \frac{t}{s^2} \right. \\
& + \frac{4}{s} + \frac{8}{3} \frac{s}{t^2} + \frac{4}{t}\bigg) + \ln(2) \ln\left(-\frac{t}{s}\right) \left(\frac{40}{9} \frac{t^2}{s^3} + \frac{92}{9} \frac{t}{s^2} + \frac{52}{3s} + \frac{64}{9} \frac{s}{t^2} \right. \\
& + \frac{116}{9t}\bigg) - \ln(2) \ln\left(-\frac{u}{s}\right) \left(\frac{40}{9} \frac{t^2}{s^3} + \frac{80}{9} \frac{t}{s^2} + \frac{40}{3s} + \frac{40}{9} \frac{s}{t^2} + \frac{80}{9t}\right) \\
& - \ln(2) \left(\frac{40}{9} \frac{t^2}{s^3} + \frac{80}{9} \frac{t}{s^2} + \frac{40}{3s} + \frac{40}{9} \frac{s}{t^2} + \frac{80}{9t}\right) - \frac{1967}{108} \frac{t^2}{s^3} - \frac{1967}{54} \frac{t}{s^2} \\
& - \frac{1967}{36s} - \frac{1967}{108} \frac{s}{t^2} - \frac{1967}{54t} \bigg\}. \tag{46}
\end{aligned}$$

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